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On the Casimir effect and the temperature inversion symmetry

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Abstract. Recently Ravndal and Lutken have discovered that the partition function for fermionic fields at finite temperature possesses a remarkable symmetry under inversion of the temperature parameter. In this communication we extend their analysis for the finite temperature Casimir effect of scalar fields. As an application of the temperature inversion symmetry idea we show how, from the Casimir energy at zero temperature, to obtain the Stefan-Boltzmann law of a Bose gas in the thermodynamic limit. We then show that in $(3+1)$ dimensions the partition function can be written in a closed form where the inversion symmetry is explicit.

1. Introduction

In 1948 Casimir [1] showed that there is an attractive force between two parallel, perfectly conducting, infinite plates due to electromagnetic field fluctuations. Since then many studies, both theoretical and experimental, have been made on this subject (see for example the review article [2]). Later studies on thermal fluctuation effects have also been included [3].

It has been noticed recently that the ratio of the free energies of bosonic and fermionic systems is the same at low and high temperature [4]. This fact has motivated some interest in the investigation of the symmetry properties of these systems under 'temperature inversion'. Recently Gundersen and Ravndal [5] showed that the scaled free energy of fermionic fields between parallel MIT-plates separated by L satisfies

$$f(\xi) = (2\xi)^4 f\left(\frac{1}{4\xi}\right) \tag{1}$$

where the function

$$f(\xi) = L^3 F(T, L) \tag{2}$$

and $\xi = LT$ are dimensionless in units where the Boltzmann constant is one. In order to clarify these results Luken and Ravndal [6] used functional methods to examine the partition function for the above system. Their results were expressed in terms of Epstein's zeta-function where the low-high temperature symmetry is easily seen.

In this paper we will examine the temperature inversion symmetry for the scalar version of the Casimir effect at finite temperature. We will consider massless scalar fields trapped inside two infinite plates separated by a distance L but otherwise free. As is well known, the vacuum energy is not a well defined quantity, being formally infinite. For massless fields, problems will appear due to the zero mode and also due to the large eigenvalues. The second kind of divergence is common and will be regularised by zeta-function techniques while the divergences associated with the zero eigenvalue will be dealt with by dimensional regularisation [7]. The regularised free energy will be shown to exhibit the 'inversion temperature symmetry'. In section 3 we illustrate these ideas with a simple application where this symmetry is used to obtain, from the Casimir energy at zero-temperature, the results of statistical mechanics for a Bose gas at high temperature (and vice versa). In section 4 we show that in four spacetime dimensions the free energy can be summed up exactly and be written in a closed form.

2. Casimir energy at finite temperature

Following [6] we will formulate this problem using functional methods. In this formalism the temperature dependence of the fields is obtained by compactification of the imaginary time direction to a circle S^1 of size β, where β is the inverse temperature. The Casimir effect is obtained by compactifying the fields in the direction perpendicular to the plates with period L. In other words we are considering the case of fields in a flat d-dimensional spacetime with the topology of $T^2 \times R^{d-2}$. A direct generalisation, the case of a hyperbox with p sides and topology $T^p \times R^{d-p}$, which is straightforwardly obtained from our results, will not be considered here. Other choices of boundary conditions may alter the value of the Casimir energy but will not influence the discussion of the symmetry under study. The partition function for the case of massless scalar fields at finite temperature can be expressed in terms of the functional determinant of the Klein–Gordon operator after one Gaussian integration

$$Z[\phi] = \int \mathrm{D}[\phi] \exp\left(-\frac{1}{2} \int \mathrm{d}x\, \phi(-\partial^2)\phi \right) = [\det(-\partial^2)]^{-1/2}. \tag{3}$$

The spectrum for the Klein–Gordon operator, in conformity with the boundary conditions, is

$$\omega_{mn}^2 = \left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{2\pi m}{L}\right)^2 + k_\mathrm{T}^2 \qquad n, m = 0, \pm 1, \pm 2, \ldots \tag{4}$$

where k_T is the momentum flowing in the transverse directions. Notice that a singularity appears due to the zero mode (ZM) of the operator, i.e., $n = m = 0$ and vanishing transverse momentum. The free energy is defined by

$$Z = \mathrm{e}^{-\beta F}. \tag{5}$$

Therefore

$$F = -\frac{1}{2\beta} \ln \det(-\partial^2). \tag{6}$$

Making use of the identity

$$\ln \det M = \mathrm{Tr} \ln M \tag{7}$$

the free energy can be expressed as

$$F = \frac{1}{2\beta} \mathrm{Tr} \ln(-\partial^2) = \frac{1}{2\beta} \sum_{m,n=-\infty}^{\infty} \int \frac{\mathrm{d}^{d-2} k_\mathrm{T}}{(2\pi)^{d-2}} \ln\left[k_\mathrm{T}^2 + \left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{2\pi m}{L}\right)^2 \right] \tag{8}$$

where one must sum over all eigenvalues. Here we are considering a d-dimensional spacetime where the plates are objects of dimension $(d-2)$. To make use of the results of dimensional regularisation we shall consider $(d-2)$ as a continuous parameter and keep n and m as integers. Making use of the DR result

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln(k^2+\alpha^2) = -\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\alpha^d \tag{9}$$

the free energy becomes

$$\begin{aligned} F &= -\frac{1}{2\beta}\frac{\Gamma[(2-d/2)]}{(4\pi)^{(d-2)/2}}\sum_{m,n}\left[\left(\frac{2\pi n}{L}\right)^2+\left(\frac{2\pi m}{\beta}\right)^2\right]^{(d-2)/2} \\ &= -\frac{1}{2\beta}\frac{\Gamma[(2-d/2)]}{(4\pi)^{(d-2)/2}}\left(\frac{2\pi}{L}\right)^{d-2}\sum_{m,n}[(n\xi)^2+m^2]^{(d-2)/2}. \end{aligned} \tag{10}$$

We are now in position to consider the ZM which, in the massless case, is an ill defined problem due to strong infrared (IR) divergences. Following the usual procedure we introduce a small mass, as a cut-off, to regulate the IR part of the integral (9). Then we obtain

$$\mathrm{ZM} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\ln(k^2+\mu_0^2). \tag{11}$$

We may evaluate the integral as we did before but must be careful to remove the IR cut-off only after we have analytically returned to the neighbourhood of d dimensions, in which the integral is IR convergent. When the operation is done in this order the ZM is seen to vanish. From now on one must understand the double summation with the term $n=m=0$ absent. The divergences due to the other values of n and m are still present and must be removed. We will make use of the Poisson summation formula (PSF) [8] to extend analytically the exponent in the sum to negative values where it is a convergent sum and at same time remove the divergent gamma function that appears in (10). Making use of the integral representation of the gamma function we have

$$\begin{aligned} \sum_{m,n}[(\eta\xi)^2+m^2]^{-z} &= \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{d}x\, x^{z-1}\sum_{m,n}\mathrm{e}^{-(n^2\xi^2+m^2)x} \\ &= \frac{\pi}{\xi\Gamma(z)}\int_0^\infty \mathrm{d}x\, x^{(1-z)-1}\sum_{m,n}\exp\left\{-\left[\left(\frac{n\pi}{\xi}\right)^2+(m\pi)^2\right]x\right\} \\ &= \frac{\pi\Gamma(1-z)}{\xi\Gamma(z)}\sum_{m,n}\left[\left(\frac{n\pi}{\xi}\right)^2+(m\pi)^2\right]^{z-1} \end{aligned}$$

where we have made use of the two-dimensional version of the PSF in the second line. Inserting this result into the free energy one gets

$$F = \frac{\Gamma(d/2)\xi^d}{2L^{d-1}\pi^{d/2}}\sum_{m,n}(n^2+m^2\xi^2)^{-d/2} = \frac{\Gamma(d/2)\xi^d}{2L^{d-1}\pi^{d/2}}Z_2\left(1,\xi,\frac{d}{2}\right) \tag{12}$$

where $Z_2(a, b, z)$ is the Epstein zeta-function [9]. One can observe that the troublesome gamma-function appearing in (10) has been cancelled. Let us introduce a function

$$f(\xi) = \frac{2L^{d-1}\pi^{d/2}}{\Gamma(d/2)}F = \xi^d Z_2\left(1,\xi,\frac{d}{2}\right). \tag{13}$$

From the definition of the Epstein zeta-function one easily sees that

$$f\left(\frac{1}{\xi}\right)=Z_2\left(\xi,1,\frac{d}{2}\right) \tag{14}$$

and from (13) and (14) we find that

$$f(\xi)=\xi^d f\left(\frac{1}{\xi}\right) \tag{15}$$

which expresses the symmetry of the system under temperature inversion. What can be seen from this formula is that for a fixed geometry the Casimir energy for massless scalar fields at zero temperature is essentially given by the free energy of a Bose gas at high temperature as expressed by the Stefan–Boltzmann law for black-body radiation [10].

3. An example

To illustrate the use of these ideas let us consider the case of a hyperbox of volume $V=L_1\ldots L_d$ at temperature β^{-1}. In the high temperature limit ($\beta \ll L_k$) or the case of a large box one can use purely dimensional arguments to write the partition function as

$$\ln Z = C(d)\frac{L_1\ldots L_d}{\beta^d} \tag{16}$$

where $C(d)$ is a number that depends on the dimension of the space and will be determined from the results of the Casimir energy. On the other hand the Casimir energy at zero temperature can be obtained from the 'inversion temperature' idea. In this situation one has $L_1 \ll \beta$, L_k; $k\neq 1$ which corresponds the case of parallel plates of area $S=L_2L_3\ldots L_d$ separated by a small distance L_1 at low temperature. The Casimir partition function is obtained from the high temperature partition function by interchanging $\beta \leftrightarrow L_1$

$$\ln Z = C(d)\frac{\beta L_2\ldots L_d}{L_1^d}. \tag{17}$$

Applying the thermodynamic relation

$$E=-\frac{1}{\beta}\ln Z \tag{18}$$

we obtain the energy per unit transverse area

$$\varepsilon_0=-\frac{C(d)}{L_1^d}. \tag{19}$$

The Casimir energy is the vacuum energy for fields confined in a bounded region

$$\varepsilon_0=\frac{1}{2}\sum_n \omega_n \tag{20}$$

and ω_n may be obtained from (4) above for large β. Then

$$\varepsilon_0=\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\mathrm{d}^{d-1}k_{\mathrm{T}}}{(2\pi)^{d-1}}\left[k_{\mathrm{T}}^2+\left(\frac{2n\pi}{L}\right)^2\right]^{1/2}. \tag{21}$$

The transverse integral can be evaluated using the results of DR

$$\int_0^\infty \frac{d^d k_T}{(2\pi)^d}\left[k_T^2+\left(\frac{2n\pi}{L}\right)^2\right]^{1/2} = -\left(\frac{n\sqrt{\pi}}{L}\right)^{d+1}\Gamma\left(\frac{-d-1}{2}\right). \tag{22}$$

Then

$$\varepsilon_0 = -\left(\frac{\sqrt{\pi}}{L}\right)^d \Gamma\left(\frac{-d}{2}\right)\zeta(-d) \tag{23}$$

where $\zeta(z)$ is the Riemann zeta-function [11]. Using the 'reflection formula', $\zeta(z)$ is analytically continued to negative values of z

$$\pi^{1-z}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sqrt{\pi}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z). \tag{24}$$

The regularised Casimir energy then becomes

$$\varepsilon_0 = -\frac{\Gamma[(d+1)/2]\zeta(d+1)}{\pi(d+1)/2}\frac{1}{L^d}. \tag{25}$$

Comparing the results (19) and (25) we arrive at

$$C(d) = \frac{\Gamma[(d+1)/2]}{\pi^{(d+1)/2}}\zeta(d+1). \tag{26}$$

In three space dimensions one gets the result

$$\ln Z = \frac{\pi^2}{90}\frac{V}{\beta^3} \tag{27}$$

where V is the volume of the box. Finally the free energy per unit volume is, using the results above, equal to

$$\frac{E}{V} = -\frac{\pi^2}{90}T^4 \tag{28}$$

which is the well known result from statistical mechanics.

4. An exact solution in four dimensions

In this section we want to show that in four spacetime dimensions one can write the scaled free energy $f(\zeta)$ in a closed form as

$$f(\xi) = \xi(1+\xi^2) \tag{29}$$

which is seen to satisfy the inversion temperature symmetry, (15), trivially. To this end we rewrite the free energy, (8), as

$$F = -\frac{1}{8\pi\beta}\int dk \sum_{m,n}\left[k^2+\left(\frac{2\pi n}{\beta}\right)^2+\left(\frac{2\pi m}{L}\right)^2\right]^{1/2}. \tag{30}$$

Extending the integral to d-continous dimensions and making use of the results of DR of the previous sections we obtain

$$F = \lim_{z \to -1, d \to 1} \left(\frac{-1}{8\pi\beta}\right) \sum_{m,n} \int \mathrm{d}^d k \left[k^2 + \left(\frac{2\pi m}{\beta}\right)^2 + \left(\frac{2\pi n}{L}\right)^2\right]^{z/2}$$
$$= \frac{\pi}{2\beta}\left(\frac{1}{\beta^2} + \frac{1}{L^2}\right)\Gamma(-1)\zeta(-2). \tag{31}$$

Making use of the reflection formula, (24), the free energy can then be written as

$$F = \frac{\zeta(3)}{4\pi L^3}\xi(1+\xi^2). \tag{32}$$

Defining the scaled free energy $f(\xi)$ as

$$f(\xi) = \frac{4\pi L^3}{\zeta(3)} F(\xi) \tag{33}$$

we obtain the desired result

$$f(\xi) = \xi(1+\xi^2). \tag{34}$$

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